Ordinals, Cardinals and Recursion[2]

Zhenfeng Tu

June 27, 2021

Purpose of The Talk

The talk will explain:

1. Definition and arithmetics of ordinals
2. Transfinite induction and recursion on ON
3. Understand cardinals and cardinality of sets.
4. Application: why do we need to count over $\mathbb{N}$?

Concept of Ordinals

What is an ordinal? It is like "ordinal number": 1st, 2nd, 135th,..., but more general than that, like $\mathbb{N}$-th, $(\mathbb{N} + 17)$-th, etc.

Definition (Ordinals)

z is an ordinal iff z is a transitive set and z is well-ordered by $\in$.

Definition (Transitive Set)

z is a transitive set iff $\forall y \in z[y \subset z]$.

We use ON to denote the class of ordinals, then

Theorem

$(ON, \in)$ is a transitive class, and ON is not a set.

Successor and Limit Ordinals

Definition (Successor of Ordinals)

Let x be an ordinal. Then the successor of x, denoted by $S(x)$, is defined as

$$S(x) = x \cup \{x\}$$

Remark

Intuitively, successor of the ordinal means "+1". For example, $1 = \{0\}$, *and* $S(1) = 1 \cup \{1\} = \{0, 1\} = 2$. *The reason we define* $+1$ *like this is to generalize "+1" operation to more general "numbers", like* $\mathbb{N} + 1$.

Successor and Limit Ordinals

Definition

An ordinal β is

- a successor ordinal iff $\beta = S(\alpha)$ for some α.
- a limit ordinal if $\beta \neq 0$ and β is not a successor ordinal.
- a finite ordinal, iff every $\alpha \leq \beta$ is either 0 or a successor.

By the axiom of infinity, set ω of all natural numbers is well defined, and is an ordinal. Furthermore, it is the least limit ordinal.

The reason ω is limit is that ω cannot be obtained by "some ordinal+1"

Arithmetics of Ordinals

We are interested in defining addition, multiplication and exponents for ordinals, as we did for natural numbers. To make a definition, we need a technical theorem.

Theorem (Ordinals-Well-ordered Sets Correspondence)

If R well-orders A, then there exists a unique $\alpha \in ON$ such that $(A, R) \cong (\alpha, \in)$. We write type$(A, R) = \alpha$.

Arithmetics of Ordinals

Sketch of proof: Apply transfinite induction.

- Let G be the set of $a \in A$ such that $\{x : x < a\}$ corresponds to some ξ. Prove the uniqueness of ξ by showing that the isomorphism between $(\alpha, <)$ and $(\beta, <)$ must be the identity map.
- Union all ξ to obtain α. Such α is unique by construction.
- WTS $G = A$. If not, then there is a least element such that its segment does not correspond to any ordinal. However, the segment of its predecessor corresponds to some ordinal γ, and $S(\gamma)$ corresponds to the segment, contradiction.
- Considering the least elements at which a statement fails and finding a contradiction is called transfinite induction.

Arithmetics of Ordinals

Intuitive explanation of theorem:

Theorem (Ordinals-Well-ordered Sets Correspondence)

If R well-orders A, then there exists a unique $\alpha \in ON$ such that $(A, R) \cong (\alpha, \in)$. We write type$(A, R) = \alpha$.

R well-orders A means:

- There is a starting point of counting, say, a.
- A is like a half line, and all elements are ordered from left to right.
- If we assume axiom of choice, we can find the "next" element for each element.

Arithmetics of Ordinals

Now if we can find next, it suffices to "count" elements of A, and give each element an ordinal number: 1st element, 2nd element,..., $\mathbb{N}$-th element, etc. α is the ordinal when we completes the counting.

The proof above is a formal justification of our intuition of counting.

Arithmetics of Ordinals

Now we can treat our ordinals as well-ordered sets, and use "type" to define addition and multiplication.

Definition (Addition, Multiplication, Exponential)

- $\alpha + \beta = \text{type}(\{0\} \times \alpha \cup \{1\} \times \beta)$
- $\alpha \cdot \beta = \text{type}(\beta \times \alpha)$
- $\alpha^0 = 1$; $\alpha^{S(\beta)} = \alpha^\beta \cdot \alpha$; $\alpha^\gamma = \sup_{\beta<\gamma}(\alpha^\beta)$, if γ is a limit ordinal. sup means the least upper bound of the set.

Arithmetics of Ordinals

Informal way of understanding addition and multiplication:

- Ordinals should be understood as "ordinal numbers", like 1st, 2st, 100th, and more generally, $\mathbb{N}$-th, $\mathbb{R}$-th. It's like you are counting a well-ordered set from its least element and following its order, and in the end you report which "number" you have counted to.
- For addition, you first count all elements in α, and then count all elements in β, and yield an answer.
- For multiplication, you count α, increase your β component by 1, and count α again, and again, until you complete the type$(\beta, \in)$-th iteration.
- Exponential is defined recursively, imitating the definition of multiplication.

Arithmetics of Ordinals

Note that ordinal addition and multiplication are not commutative, not right-distributive, and has no right cancellation laws. For example:

- $1 + \omega = \omega$; $\omega + 1 > \omega$
- $(1 + 1) \cdot\omega < \omega + \omega$.

For $(1 + 1) \cdot \omega$, we are counting "2" for ω times—this will never bring us over ω. However, $\omega + \omega$ means "we first count ω, following which we count yet another ω"—this is "more than" ω)

Arithmetics of Ordinals

To prove the relations, use the following lemma:

Lemma

If R well-orders A and $X \subset A$, then R well-orders X and $type(X, R) \leq type(A, R)$.

The proof of lemma is as follows. WLOG assume A is an ordinal. There is an isomorphism from X to some ordinal δ preserving the relation. Let f be the isomorphism and by primitive induction on ordinals we can show $f(\xi) < \xi$.

After proving the lemma, we can prove the ordering. The associative laws follows from $(\alpha + \beta) + \gamma \leq \alpha + (\beta + \gamma)$ and $(\alpha + \beta) + \gamma \geq \alpha + (\beta + \gamma)$. Other laws are similar.

Induction and Recursion

In the world of natural numbers, we have induction. Now we want to extend the idea of induction to ordinals. Usually, we prove $\forall \xi < \alpha, \varphi(\xi)$ by deriving a contradiction from the least $\xi < \alpha$ such that $\not\varphi(\xi)$.

Theorem (Transfinite Induction on ON)

For each formula $\psi(\alpha)$: if $\psi(\alpha)$ holds for some ordinal α, then there exists a least ordinal ξ such that $\psi(\xi)$.

Proof: If α least, done. If α not least, then $X = \{\xi < \alpha : \psi(\xi)\}$ is non-empty and well-ordered by $\in$, so it has a least element.

Induction and Recursion

In the world of natural numbers, we have recursively defined sequence like Fibonacci sequence. We want to extend the idea of recursive definition to defining "sequences" indexed by ordinals.

Induction and Recursion

Take Fibonacci numbers as an example: $f(0) = f(1) = 1$, $f(x) = f(x-1) + f(x-2)$ when $x \geq 2$. Intuitively, the recursive definition gives rise to a function from natural numbers to natural numbers, with $n \mapsto f(n)$.

The general case, however, is not clear (at least for now). For example, we have defined exponential of ordinals recursively:

$$\alpha^0 = 1, \alpha^{S(\beta)} = \alpha^\beta \cdot \alpha, \alpha^\gamma = \sup_{\beta<\gamma} \alpha^\beta \text{ if } \gamma \text{ is a limit ordinal.}$$

For this definition however, we don't know whether this definition indeed defines α^ξ for each ordinal ξ. Or equivalently, we don't know whether the definition gives rise to a function F on each set of ordinals such that F satisfies the recursive relation.

Induction and Recursion

The theorem below guarantees that it is legitimate to recursively define a "function" on ON recursively.

Theorem (Primitive Recursion on ON)

Suppose that $\forall s \exists ! y \varphi(s, y)$, and define $G(s)$ to be the unique y such that $\varphi(s, y)$. Then we can define a formula ψ for which the following are provable:

- $\forall x \exists ! y \psi(x, y)$, *so ψ defines a function F where $F(x)$ is the y such taht $\psi(x, y)$.*
- $\forall \xi \in ON$, $F(\xi) = G(F|\xi)$.

Informally, this theorem states that if we have a recursive relation on ON, then it defines a function that satisfies the relation.

Induction and Recursion

Let's see how we generalize to the statement above step by step.

Theorem (Primitive Recursion on ON, Informal)

Suppose there is a natural number s_ξ associated to each element ξ of δ, where δ is an ordinal. The numbers are such that $s_\xi = G(s_{\xi-1})$ if $\xi - 1$ exists, and $s_\xi = 0$ if $\xi - 1$ does not exist . Then the recurrence relation defines a function F from δ to ω.

Now this is intuitively true because "you know how to get to the next number and the next number is uniquely determined."

Now generalization: In the step above ξ depends only on its previous element. How about just general dependence on everything before?

Induction and Recursion

For this purpose, we use $s|\delta$ to denote δ-tuple $(s_0, s_1, \ldots, s_\alpha, \ldots,)_{\alpha<\delta}$.

Theorem (Primitive Recursion on ON)

Suppose there is a set s_ξ associated to each element ξ of δ, where δ is an ordinal. The numbers are such that $s_\xi = G(s|\xi)$. Then the recurrence relation defines a function F from δ to ω such that

- $\forall \xi \in \delta \cap ON$, $F(\xi) = G(F|\xi)$

Generalization: we are restricting ourselves to a specific set δ. How about something for ON, which is too large to be a set?

Induction and Recursion

Now to work with a proper class, we can generalize functions to the following form of statements: For example, if F is a "function" defined on all $\delta \in ON$, then $\forall x \in ON \exists! y \psi(x, y)$ where $\psi(x, y) = (y = F(x))$ contains exactly the same information as F; while functions can't have ON as its domain, the expression is "defined" on ON.

Also we have to modify G similarly. G was defined δ-tuples; ON-tuples is not a set. We use similar formula replacement as above.

To get its most general form, we can set uninteresting values of φ and ψ to some fixed values. So now we get our desired theorem:

Theorem (Primitive Recursion on ON)

Suppose that $\forall s \exists! y \varphi(s, y)$, and define $G(s)$ to be the unique y such that $\varphi(s, y)$. Then we can define a formula ψ for which the following are provable:

- $\forall x \exists! y \psi(x, y)$, *so ψ defines a function F where $F(x)$ is the y such taht $\psi(x, y)$.*
- $\forall \xi \in ON,\ F(\xi) = G(F|\xi)$.

Induction and Recursion

Proof of the theorem:

- Restrict ourselves to some sets: for each $\delta \in ON$, prove that there exists some h_δ defined on $\delta \in ON$ such that it satisfies the recursive relation. Existence follows from transfinite induction.
- We must show the h_δ and h_γ agree on the domain where they are both defined. This also follows from transfinite induction.
- Now each $x \in ON$ will be mapped to some y by h and the y is independent of the choice of h. So we can define ψ to be such that
  - it gives the corresponding y for $x \in ON$;
  - it gives 0 for $x \notin ON$.

  So F agrees with h by definition of ψ.

Induction and Recursion

As a reward for our hard work, we can now define generalized sequence:

Definition (α-sequence)

For $\alpha \in ON$, an α-sequence is a function s with domain α, and $s_\xi = s(\xi)$ for $\xi < \alpha$.

Power Sets

Definition

The power set of x is $\mathcal{P}(x) = \{z : z \subset x\}$.

Definition

$B^A = {}^A B$ is the set of function with $\mathrm{dom}(f) = A$ and $\mathrm{ran}(f) \subset B$.

Definition

$A^{<\alpha} = {}^{<\alpha}A = \cup_{\xi<\alpha} A^\xi$.

This defines the set of all sequences with length at most α and elements in A.

Cardinality and Cardinals

Our next topic would be cardinality and cardinals. We want to understand the how many elements are there in some set.

For finite sets, it suffices to count the number of elements, and the number is independent of the way you count. For infinite sets however, the ordinal you obtain depends on the way you count. For example, we can count from 2, and after counting all the way to the end, we count 1. This way we conclude that there are $\omega + 1$ natural numbers. Similarly, by counting from n, we can conclude that there are $\omega + n$ natural numbers.

Therefore the naive counting is not consistent for infinite sets. Now, we define two sets are of the same size iff there is a bijection between them.

Cardinality and Cardinals

The formal definition is the following:

Definition

$X \preceq Y$ iff there is an injective function $f : X \to Y$; $X \cong Y$ iff there is a bijective function $f : X \to Y$.

However, finding bijections is usually difficult (try to find a bijection from $[0, 1]$ to $[0, 1]^2$!). To prove that there exists a bijection between two sets, we usually use

Theorem (Schröder-Bernstein Theorem)

$A \cong B$ iff $A \preceq B$ and $B \preceq A$.

Carnality and Cardinals

First, look at a lemma.

Lemma

If $B \subset A$ and $f : A \to B$ is injective then $A \cong B$.

Proof:We want to show f is surjective. Notice:

$$A \supset B \supset f^1(A) \supset f^1(B) \supset \dots$$

So $P = \cap_{n<\omega} f^n(A) = \cap_{n<\omega} f^n(B)$. Let $H_n = f^n(A) \setminus f^n(B)$, $K_n = f^n(B) \setminus f^{n+1}(A)$. So f maps H_n and K_n bijectively to H_{n+1} and K_{n+1}. H_i and H_j are disjoint if $i \neq j$. Because of the following decomposition,

$$A = P \cup H_0 \cup H_1 \dots \cup K_0 \cup K_1 \dots$$

$$B = P \cup H_1 \cup H_2 \dots \cup K_0 \cup K_1 \dots$$

Therefore f is surjective.

Cardinality and Cardinals

The Schröder-Bernstein Theorem follows easily from the lemma. Using the theorem we easily conclude that

- ${}^{A}2 = \mathcal{P}(A)$
- If $A \preceq B$ and $C \preceq D$ then ${}^{A}C \preceq {}^{B}D$.

The first follows by associating a set with its characteristic function, and the second follows by considering diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \\
C \arrow[r] & D
\end{tikzcd}$$

Cardinality and Cardinals

We notice that the ordinals comes in blocks of the same size: same size means we can find a bijection between them. For example, ω, $\omega + 1, \ldots, \omega + \omega$ are all of the same size. To express the size of ordinals, we have the following definition:

Definition (Cardinal)

A cardinal is an ordinal α such that $\xi \prec \alpha$ for all $\xi < \alpha$

Definition (Finite and Countable)

α is countable iff $\alpha \preceq \omega$. α is finite if $\alpha \preceq n$ for some natural number n.

Cardinality and Cardinals

Now we are finally in a position to define the cardinality of sets:

Definition (Cardinality)

If A is well-orderable, then $|A|$ is the least ordinal α such that $A \approx \alpha$.

Application of Ordinals

We extended properties of natural numbers to ordinals, and got a quite satisfactory theory. But why do we want to count over $\mathbb{N}$, instead of stopping at $\mathbb{N}$, as we did in elementary school? Let's see an example.

Warning: This application is intended for those familiar with real analysis or measure theory. It is purely optional, and you may safely ignore this part without affecting your study of later sections.

Application of Ordinals

Let $\mathcal{E}$ be a family of subsets in X, and $\mathcal{M}(\mathcal{E})$ be the smallest σ-algebra that contains $\mathcal{E}$. We say $\mathcal{M}(\mathcal{E})$ is the σ-algebra generated by $\mathcal{E}$, and our target is to construct the smallest σ-algebra containing $\mathcal{E}$.

Now, we define

$$\mathcal{E}_1 = \mathcal{E} \cup \{A^C : A \in \mathcal{E}\}$$

and recursively,

$$\mathcal{E}_j = \{\cup_i A_i : i \in \mathbb{N}, A_i \in \mathcal{E}_{j-1}\} \cup \{(\cup_i A_i)^C : i \in \mathbb{N}, A_i \in \mathcal{E}_{j-1}\}$$

Each step we add countable union and countable intersection of

sets of our family to make our new and larger family of sets.

Application of Ordinals

Lemma

We claim that

$$\mathcal{M}(\mathcal{E}) = \cup_{\alpha \in \Omega} \mathcal{E}_\alpha$$

where Ω is the set of all countable ordinals. [1]

Why not union over all natural numbers? It's because if $E_i \in \mathcal{E}_{i+1} \setminus \mathcal{E}_i$, then $\cup_i E_i$ has no reason to be in $\cup_i \mathcal{E}_i$. While it might be possible to choose $E_\alpha \in \mathcal{E}_{\alpha+1} \setminus \mathcal{E}_\alpha$, it is okay if $\cup_{\alpha \in \Omega} E_\alpha$ is not an element of $\mathcal{M}(\mathcal{E})$ because this is an uncountable union.

We need to count "over" $\mathbb{N}$ to construct $\mathcal{M}(\mathcal{E})$!

Application of Ordinals

Proof: First we show $\cup_{\alpha\in\Omega}\mathcal{E}_\alpha \subset \mathcal{M}(\mathcal{E})$. Use transfinite induction. Obviously $\mathcal{E}_1 \subset \mathcal{M}(\mathcal{E})$. Assume ξ is the least ordinal such that $\mathcal{E}_\xi$ is not a subset of $\mathcal{M}(\mathcal{E})$. So $\mathcal{E}_{\xi-1} \subset \mathcal{M}(\mathcal{E})$, and since $\mathcal{M}(\mathcal{E})$ is closed under countable union and complements, $\mathcal{E}_\xi \subset \mathcal{M}(\mathcal{E})$, contradiction. This proves that $\cup_{\alpha\in\Omega} \subset \mathcal{M}(\mathcal{E})$

Application of Ordinals

Proof: Next we show that $\mathcal{M}(\mathcal{E}) \subset \cup_{\alpha \in \Omega} \mathcal{E}_\alpha$. Since $\mathcal{M}(\mathcal{E})$ is defined to be the smallest σ-algebra containing $\mathcal{E}$, it suffices to show $\cup_{\alpha \in \Omega} \mathcal{E}_\alpha$ is a σ-algebra containing $\mathcal{E}$. Let $E_j \in \mathcal{E}_{\alpha_j}$. Therefore, $\forall j$, $E_j \in \mathcal{E}_{\sup \alpha_j}$, and $\cup_j E_j \in \mathcal{E}_{\sup \alpha_j + 1}$. This completes the proof.

Summary

We defined ordinals, and transplant everything we know about natural numbers to ordinals: we can now:

- Add two ordinals
- Multiply two ordinals
- Raising an ordinal to the power of some other ordinal
- Do induction in ON
- Recursively define "sequences" indexed by ordinals

Summary

We proved basic theorems for transfinite induction and primitive recursion on ON. More specifically, we

- proved a theorem scheme: transfinite induction on ON.
- defined α-sequence for every $\alpha \in ON$.
- constructed σ-algebra generated by every family of subsets.

We also defined power sets, cardinals and cardinality of sets, and proved Schröder-Bernstein theorem.

References

[1] G. B. Folland. *Real analysis: modern techniques and their applications*, volume 40. John Wiley & Sons, 1999.

[2] K. Kunen. The foundations of mathematics, 2007.